

THE GRAM DETERMINANT OF THE TYPE B TEMPERLEY-LIEB ALGEBRA

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1. INTRODUCTION

In this paper, we solve a problem posed by the late Rodica Simion regarding type B Gram determinants, cf. [7]. We present this in a fashion influenced by the work of W.B.R. Lickorish on Witten-Reshetikhin-Turaev invariants of 3-manifolds. We will give a history of this problem in a sequel paper in which we also plan to address other related questions by Simion [8, 7] and connect the problem to Frenkel-Khovanov's work [2].

2. THE TYPE B GRAM DETERMINANT

Let \mathbf{A}_n be an annulus with $2n$ points, a_1, \dots, a_{2n} , on the outer circle of the boundary, cf. Fig 1. Let $\mathbf{b}_n := \{b_1, b_2, \dots, b_{\binom{2n}{n}}\}$ be the set of all possible diagrams,

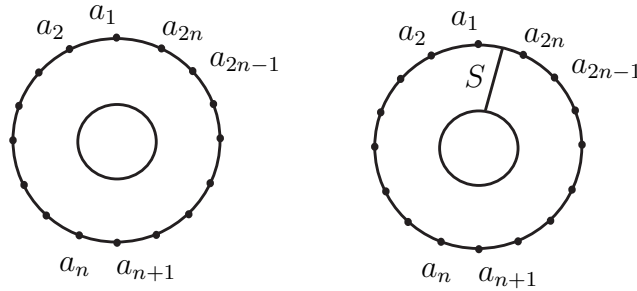


FIGURE 1. \mathbf{A}_n and \mathbf{A}_n with a segment S

up to deformation, in \mathbf{A}_n with n non-crossing chords connecting these $2n$ points, Fig 2. We define a pairing $\langle \cdot, \cdot \rangle$ on \mathbf{b}_n as follows: Given $b_i, b_j \in \mathbf{b}_n$ we glue

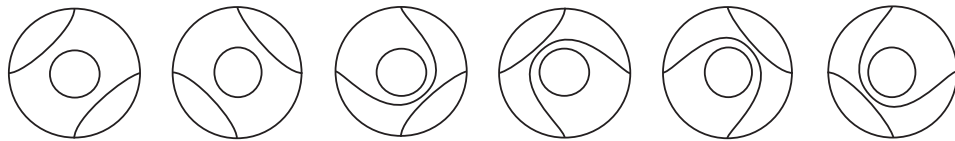


FIGURE 2. Connections in $\mathbf{b}_2 = \{b_1, b_2, b_3, b_4, b_5\}$

b_i with the inversion of b_j along the marked circle, respecting the labels of the marked points. The resulting picture is an annulus with two types of disjoint circles, homotopically non-trivial and trivial; compare Fig 3. The bilinear form $\langle \cdot, \cdot \rangle$ is defined by $\langle b_i, b_j \rangle = \alpha^m \delta^n$ where m and n are the number of homotopically non-trivial circles and homotopically trivial circles respectively.

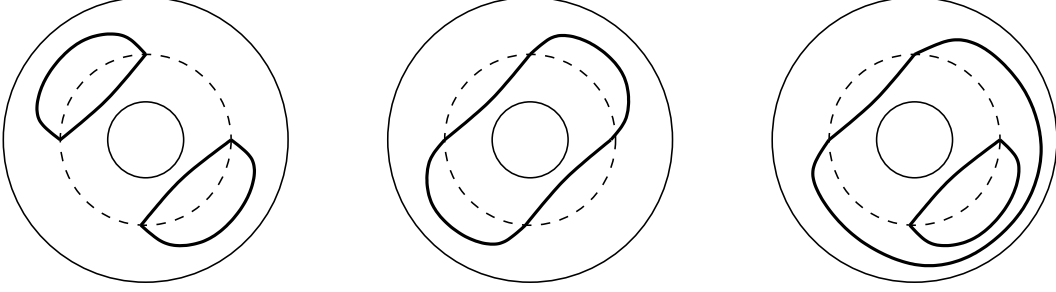


FIGURE 3. $\langle b_1, b_1 \rangle = \delta^2$, $\langle b_1, b_2 \rangle = \alpha$, $\langle b_1, b_3 \rangle = \alpha\delta$; $b_i \in \mathbf{b}_2$

Let

$$G_n(\alpha, \delta) = \left(\langle b_i, b_j \rangle \right)_{1 \leq i, j \leq \binom{2n}{n}}$$

be the matrix of the pairing on \mathbf{b}_n called the Gram matrix of the type B Temperley-Lieb algebra. We denote its determinant by $D_n^B(\alpha, \delta)$. The roots of $D_n^B(\alpha, \delta)$ were predicted by Dąbkowski and Przytycki, and the complete factorization of $D_n^B(\alpha, \delta)$ was conjectured by G.Barad:

Conjecture 1 (G. Barad).

$$D_n^B(\alpha, \delta) = \prod_{i=1}^n (T_i(\delta)^2 - \alpha^2)^{\binom{2n}{n-i}}$$

where $T_i(\delta)$ is the Chebyshev (Tchebycheff) polynomial of the first kind:

$$T_0 = 2, \quad T_1 = \delta, \quad T_i = \delta T_{i-1} - T_{i-2}.$$

The rest of the paper is devoted to a proof of Conjecture 1. It follows directly from the following two lemmas, the first of which is proven in Section 3.

Lemma 1. For $i \geq 1$, $\alpha = (-1)^{i-1} T_i(\delta)$ is a zero of $D_n^B(\alpha, \delta)$ of multiplicity at least $\binom{2n}{n-i}$.

Lemma 2. Let S be a line segment connecting the two boundary components of \mathbf{A}_n such that S is disjoint from a_i , $1 \leq i \leq 2n$; see 1. Let $c(b_i)$ denote the number of chords in b_i that cut S , and let $P = (p_{ij})$ be a diagonal matrix defined by $p_{ii} = (-1)^{c(b_i)}$. Then $G_n(-\alpha, \delta) = P G_n(\alpha, \delta) P^{-1}$.

Proof. The power of α in $\langle b_i, b_j \rangle$ is congruent to $c(b_i) + c(b_j)$ modulo 2, thus

$$\langle b_i, b_j \rangle|_{\alpha \mapsto -\alpha} = (-1)^{c(b_i)+c(b_j)} \langle b_i, b_j \rangle$$

and Lemma 2 follows. \square

Proof of Conjecture 1. According to Lemma 2, $G_n(-\alpha, \delta)$ and $G_n(\alpha, \delta)$ are conjugate matrices. Hence $\alpha = (-1)^i T_i(\delta)$ is a zero of $D_n^B(\alpha, \delta)$ of the same multiplicity as $\alpha = (-1)^{i-1} T_i(\delta)$. Therefore, by this and Lemma 1 we have

$$D_n^B(\alpha, \delta) = p \prod_{i=1}^n (T_i(\delta)^2 - \alpha^2)^{\binom{2n}{n-i}},$$

for some $p \in \mathbb{Z}[\alpha, \delta]$. The diagonal entries in $G_n(\alpha, \delta)$ are all equal to δ^n and they are of highest degree in each row thus $D_n^B(\alpha, \delta)$ is a monic polynomial in variable δ of degree $n \binom{2n}{n}$. Furthermore $T_i(\delta)$ is a monic polynomial of degree i . Couple these with a well known equality[†],

$$2 \sum_{i=1}^n i \binom{2n}{n-i} = n \binom{2n}{n},$$

to conclude that $p = 1$. \square

3. PROOF OF LEMMA 1

It is enough to show that the nullity of $G_n((-1)^{i-1} T_i(\delta), \delta)$ is at least $\binom{2n}{n-i}$, which we prove by the theory of Kauffman Bracket Skein Module (KBSM); see [3, 6] for the definition and properties of KBSM. Denote the KBSM of a 3-manifold X by $\mathcal{S}(X)$. Let \mathbf{A} be an annulus. For any two elements x, y in $\mathcal{S}(\mathbf{A}) = \mathbb{Z}[A^{\pm 1}, \alpha]^{\ddagger}$ let $H(x, y)$ be the element in $\mathcal{S}(S^3) = \mathbb{Z}[A, A^{-1}]$ obtained by decorating the two components of the Hopf link with x and y . Denote the k -th Jones-Wenzl idempotent by f_k , cf. [5]. Define a linear map

$$\phi_k : \mathcal{S}(\mathbf{A}) \rightarrow \mathbb{Z}[A, A^{-1}]$$

such that $\phi_k(x) = H(x, \hat{f}_k)$, where $\hat{f}_k \in \mathcal{S}(\mathbf{A})$ is the natural closure of f_k . For $b_i, b_j \in \mathbf{b}_n$, we will consider $\langle b_i, b_j \rangle$ as an element of $\mathcal{S}(\mathbf{A})$. If $\langle b_i, b_j \rangle = \alpha^m \delta^n$ then

$$\phi_k(\langle b_i, b_j \rangle) = (-A^{2(k+1)} - A^{-2(k+1)})^m (-A^2 - A^{-2})^n \Delta_k, \quad (1)$$

[†]We use “telescoping” to get $2 \sum_{i=1}^n i \binom{2n}{n-i} = 2 \sum_{i=1}^n n \left(\binom{2n-1}{n-i} - \binom{2n-1}{n-i-1} \right) = 2n \binom{2n-1}{n-1} = n \binom{2n}{n}$.

[‡]For any surface F , we write $\mathcal{S}(F)$ for $\mathcal{S}(F \times [0, 1])$. In $\mathcal{S}(\mathbf{A})$, α represents a nontrivial curve, 1 – the empty curve, and $\delta = -A^2 - A^{-2}$ – a trivial curve. \mathbf{b}_n is a basis of a relative KBSM, $\mathcal{S}(\mathbf{A}_n)$ as a module over $\mathbb{Z}[A^{\pm 1}, \alpha]$.

where $\Delta_k = (-1)^k(A^{2(k+1)} - A^{-2(k+1)})/(A^2 - A^{-2})$ is the Kauffman bracket of \hat{f}_k ; see page 143 of [5]. To relate the Gram matrix $G_n(\alpha, \delta)$ to the map ϕ_k we substitute $\delta = -A^2 - A^{-2}$ to obtain $T_k(\delta) = (-1)^k(A^{2k} + A^{-2k})$. Let

$$F_{n,k} = \left(\phi_{k-1}(\langle b_i, b_j \rangle) \right)_{1 \leq i, j \leq \binom{2n}{n}}.$$

Then

$$G_n((-1)^{k-1}T_k(-A^2 - A^{-2}), -A^2 - A^{-2}) = \frac{1}{\Delta_{k-1}} F_{n,k}.$$

Therefore, Lemma 1 follows from the next lemma.

Lemma 3. *The nullity of $F_{n,k}$ is at least $\binom{2n}{n-k}$.*

To prove Lemma 3 we need some linear maps defined on $\mathcal{S}(\mathbf{A}_n)$ and $\mathcal{S}(\mathbf{D}_{n,k})$, $k \geq 0$, where $\mathbf{D}_{n,k}$ is the disk with $2(n+k)$ points on its boundary. These points are labeled counter-clockwise by a_1, \dots, a_{2n} , l_1, \dots, l_k and u_k, \dots, u_1 .

Let

$$\psi_{n,k} : \mathcal{S}(\mathbf{A}_n) \rightarrow \mathcal{S}(\mathbf{D}_{n,k})$$

be a linear map defined as follows: Let $\psi'_{n,k}$ be an embedding of \mathbf{A}_n into a neighborhood of the boundary of $\mathbf{D}_{n,k}$ such that the point a_i on \mathbf{A}_n is mapped to a_i . Let $L \subset \mathbf{D}_{n,k}$ denote the lollipop consisting of the image of the inside boundary of \mathbf{A}_n together with a line segment connecting it to a point between u_k and l_k . If $[x]$ is a diagram representing an element $x \in \mathcal{S}(\mathbf{A}_n)$ then $\psi_{n,k}(x)$ is represented by a diagram in $\mathbf{D}_{n,k}$ consisting of $\psi'_{n,k}[x]$ and k chords in $\mathbf{D}_{n,k} \setminus L$ parallel to L such that if a segment of $\psi'_{n,k}[x]$ intersects L then it intersects the k parallel chords in k over-crossings above the lollipop stick and k under-crossings beneath the stick. See Fig 4 for a value of $\psi_{3,2}$. We also need a linear map

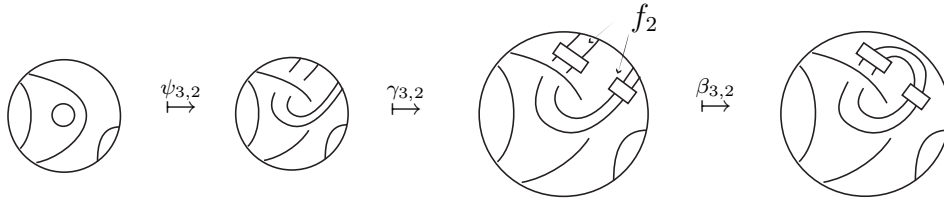


FIGURE 4.

$$\gamma_{n,k} : \mathcal{S}(\mathbf{D}_{n,k}) \rightarrow \mathcal{S}(\mathbf{D}_{n,k})$$

defined by inserting 2 copies of the Jones-Wenzl idempotents f_k close to u_1, \dots, u_k and l_1, \dots, l_k . See Fig 4 for a value of $\gamma_{3,2}$. The third map

$$\beta_{n,k} : \mathcal{S}(\mathbf{D}_{n,k}) \rightarrow \mathcal{S}(\mathbf{D}_{n,0})$$

is defined by connecting u_i to l_i outside $\mathbf{D}_{n,k}$ and then pushing these arcs into $\mathbf{D}_{n,k}$. See Fig 4 for a value of $\beta_{3,2}$. The fourth map

$$\zeta_n : \mathcal{S}(\mathbf{D}_{n,0}) \times \mathcal{S}(\mathbf{A}_n) \rightarrow \mathcal{S}(\mathbf{D}_{0,0})$$

is defined by gluing the two entries according to the marks. It is clear that for all $x, y \in \mathbf{A}_n$ we have

$$\phi_k(\langle x, y \rangle) = \zeta_n(\beta_{n,k} \circ \gamma_{n,k} \circ \psi_{n,k}(x), y). \quad (2)$$

Therefore, if one can show that $\gamma_{n,k-1} \circ \psi_{n,k-1}(b_{i_j})$, $1 \leq j \leq s$ for some integer s , are linearly dependent then so are the corresponding rows in $F_{n,k}$. This observation is used to prove Lemma 3.

Proof of Lemma 3. By the above argument it is enough to show that $\gamma_{n,k-1} \circ \psi_{n,k-1}(\mathbf{b}_n)$ is contained in a subspace of dimension $\binom{2n}{n} - \binom{2n}{n-k}$ in $\mathcal{S}(\mathbf{D}_{n,k-1})$. Therefore, it suffices to show that

$$\dim(\text{Im}(\gamma_{n,k-1})) \leq \binom{2n}{n} - \binom{2n}{n-k}. \quad (3)$$

Let $\text{NC}(\mathbf{D}_{n,k-1})$ be the set of non-crossing diagrams in $\mathbf{D}_{n,k-1}$ consisting of $n+k-1$ chords. Then $\text{NC}(\mathbf{D}_{n,k-1})$ is a basis of $\mathcal{S}(\mathbf{D}_{n,k-1})$. If $x \in \text{NC}(\mathbf{D}_{n,k-1})$ contains a chord connecting two u_i 's or two l_i 's then $\gamma_{n,k-1}(x) = 0$ by a well known property of the Jones-Wenzl idempotent, cf. Lemma 13.2 in [5]. Hence this lemma follows from the inequality

$$|\tilde{\text{NC}}(\mathbf{D}_{n,k-1})| \leq \binom{2n}{n} - \binom{2n}{n-k}$$

where $\tilde{\text{NC}}(\mathbf{D}_{n,k-1})$ is the set of diagrams in $\text{NC}(\mathbf{D}_{n,k-1})$ with no chord connecting two u_i 's or two l_i 's. In fact, the equality holds: \square

Lemma 4. *Assume the notation above. We have*

$$|\tilde{\text{NC}}(\mathbf{D}_{n,k})| = \binom{2n}{n} - \binom{2n}{n-k-1}. \quad (4)$$

Proof. Lemma 4 is a standard combinatorial fact[§] but we give its proof for completeness. Recall that \mathbf{A}_n denotes an annulus with $2n$ marks, labeled a_1, \dots, a_{2n} , on the outer boundary component. Fix a point x_0 between a_{2n} and a_1 on the marked boundary circle such that the arc containing x_0 has no other a_i 's. Let S be a line segment connecting x_0 to the other boundary component of \mathbf{A}_n , see 1.. Let $\text{NC}_{\leq k}(\mathbf{A}_n) \subset \mathbf{b}_n$ be the set of non-crossing diagrams in \mathbf{A}_n consisting of n chords which intersect S at most k times. There is a 1-1 correspondence between $\text{NC}_{\leq k}(\mathbf{A}_n)$ and $\tilde{\text{NC}}(\mathbf{D}_{n,k})$. (Suppose $x \in \text{NC}_{\leq k}(\mathbf{A}_n)$ intersects S at k' times. Draw $k - k'$ circles close and parallel to the unmarked boundary component of \mathbf{A}_n . Cut along S and we obtain a diagram in $\tilde{\text{NC}}(\mathbf{D}_{n,k})$.)

Hence it is enough to show that the set $\text{NC}_{\geq j}(\mathbf{A}_n) := \mathbf{b}_n \setminus \text{NC}_{\leq j-1}(\mathbf{A}_n)$ has $\binom{2n}{n-j}$ elements. We construct a bijection between $\text{NC}_{\geq j}(\mathbf{A}_n)$ and the choices of $n-j$ marks among the $2n$ marks on \mathbf{A}_n .

[§]It can be derived from the reflection principle by Desiré André, 1887; also compare [4, 9, 10, 1].

(i) Assume $n - j$ marks, $a_{i_1}, \dots, a_{i_{n-j}}$, are chosen. We construct n chords as follows: If, for some s , the point a_{i_s+1} is not chosen, we draw an *oriented* chord from a_{i_s} to a_{i_s+1} , not cutting S . If a_{2n} is chosen but a_1 is not, then draw an oriented chord from a_{2n} to a_1 cutting S . At this stage at least one chord is drawn. Delete this chord together with its endpoint marks and repeat the process again until all chosen marks are used (they are the beginning marks of the constructed chords). We are left with $2j$ marks. Choose the mark with the largest index and draw a chord as before. All new j chords cut S so the constructed diagram cuts S in at least j points.

(ii) Conversely consider a diagram of n chords cutting S at least j times. Orient these chords counter-clockwise. Among the n chords there are $s \leq n - j$ of them not cutting S . Add to these s chords $n - j - s$ more chords which are as close to the outside circle of \mathbf{A}_n as possible. The beginning marks of these $n - j$ chords are the marks corresponding to our diagram.

This ends the construction of the bijection. Hence we have
 $|\text{NC}_{\geq j}(\mathbf{A}_n)| = \binom{2n}{n-j}.$ □

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